Universität des Saarlandes
Fachrichtung 7.1 – Theoretische Physik
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Übungen zur Theoretischen Physik III für Lehramtskandidaten
(Quantenmechanik+Statistische Physik)
SS 2010, Blatt 1

1. Aufgabe: Wellenfunktion (3+3 Punkte)

Einige Regeln für komplexe Konjugation:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{\varphi(z)} = \varphi(\overline{z}) \text{ für jede holomorphe Funktion } \varphi \text{ mit } \varphi(x) \in \mathbb{R} \text{ für } x \in \mathbb{R}$$

a)

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{\infty} dk\ g(k) e^{i\left(kx - \hbar\frac{k^2}{2m}t\right)} \\
&= \int_{-\infty}^{\infty} dk\ A e^{-\frac{a^2}{2}k^2} e^{i\left(kx - \hbar\frac{k^2}{2m}t\right)} \\
&= A \int_{-\infty}^{\infty} dk\ e^{ikx} e^{-\left(\frac{a^2}{2} + i\frac{\hbar t}{2m}\right)k^2} \\
&= A\frac{\sqrt{2\pi}}{a}\left(1 + i\frac{\hbar t}{ma^2}\right)^{-1/2} \exp\left[-\frac{x^2}{2a^2\left(1 + i\frac{\hbar t}{ma^2}\right)}\right]
\end{aligned}$$

Komplexe Konjugation der Wellenfunktion ergibt:

$$\begin{aligned}
\psi^*(x,t) &= A^*\frac{\sqrt{2\pi}}{a}\left(\left(1 + i\frac{\hbar t}{ma^2}\right)^{-1/2}\right)^* \left(\exp\left[-\frac{x^2}{2a^2\left(1 + i\frac{\hbar t}{ma^2}\right)}\right]\right)^* \\
&= A^*\frac{\sqrt{2\pi}}{a}\left(1 - i\frac{\hbar t}{ma^2}\right)^{-1/2} \exp\left[-\frac{x^2}{2a^2\left(1 - i\frac{\hbar t}{ma^2}\right)}\right]
\end{aligned}$$

$$\begin{aligned}
|\psi(x,t)|^2 &= \psi(x,t)\psi^*(x,t) \\
&= |A|^2\frac{2\pi}{a^2}\left[\left(1 + i\frac{\hbar t}{ma^2}\right)\left(1 - i\frac{\hbar t}{ma^2}\right)\right]^{-1/2} \exp\left[-\frac{x^2}{2a^2}\left(\frac{1}{1 + i\frac{\hbar t}{ma^2}} + \frac{1}{1 - i\frac{\hbar t}{ma^2}}\right)\right] \\
&= |A|^2\ \frac{2\pi}{a^2}\left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]^{-1/2} \exp\left[-\frac{x^2}{a^2\left(1 + \left(\frac{\hbar t}{ma^2}\right)^2\right)}\right]
\end{aligned}$$

Normierung:

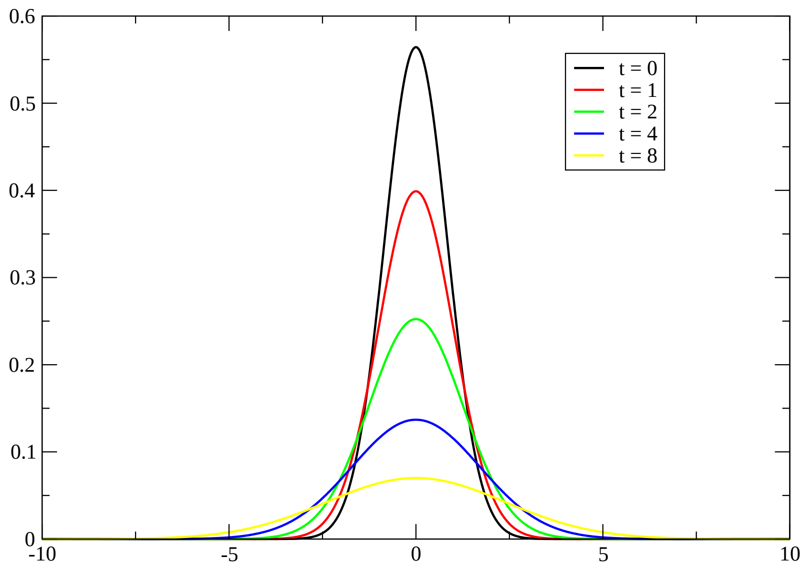
$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} dx \;\; |\psi(x,t)|^2 = |A|^2 \, \frac{2\pi}{a^2} \left[1 + \left(\frac{\hbar t}{m a^2}\right)^2\right]^{-1/2} \left(2\pi \frac{a^2}{2}\left(1 + \left(\frac{\hbar t}{m a^2}\right)^2\right)\right)^{1/2} \\
&= |A|^2 \frac{2\pi^{3/2}}{a} \\
\Rightarrow A &= \left(\frac{a}{2\pi^{3/2}}\right)^{1/2}
\end{aligned}$$

Damit lautet die Wellenfunktion:

$$\psi(x,t) = a^{-1/2}\pi^{-1/4}\left(1 + i\frac{\hbar t}{m a^2}\right)^{-1/2} \exp\left[-\frac{x^2}{2a^2\left(1 + i\frac{\hbar t}{m a^2}\right)}\right]$$

b)

$$|\psi(x,t)|^2 = \frac{1}{a\sqrt{\pi}}\left[1 + \left(\frac{\hbar t}{m a^2}\right)^2\right]^{-1/2} \exp\left[-\frac{x^2}{a^2\left(1 + \left(\frac{\hbar t}{m a^2}\right)^2\right)}\right]$$



Setze

$$\sigma^2 := \frac{a^2}{2}\left(1 + \left(\frac{\hbar t}{m a^2}\right)^2\right).$$

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} dx\ \psi^*(x,t) x \psi(x,t) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\ x \exp\left[-\frac{x^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\ \frac{d}{dx}\left[-\sigma^2 \exp\left[-\frac{x^2}{2\sigma^2}\right]\right] \\
&= -\frac{\sigma}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{C}\right]\Bigg|_{-\infty}^{+\infty} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} dx\ \psi^*(x,t) x^2 \psi(x,t) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\ x \cdot x \exp\left[-\frac{x^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(x\left(-\sigma^2\right) \exp\left[-\frac{x^2}{2\sigma^2}\right]\Bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} dx\ \left(-\sigma^2\right) \exp\left[-\frac{x^2}{2\sigma^2}\right] \right) \\
&= \sigma^2
\end{aligned}$$

$$\begin{aligned}
\Delta x &= \sqrt{\langle (x - \langle x \rangle) \rangle} = \sqrt{\langle x^2 \rangle} \\
&= \sigma \\
&= \frac{a^2}{2}\left(1 + \left(\frac{\hbar t}{m a^2}\right)^2\right)
\end{aligned}$$

2. Aufgabe: Ehrenfest'sche Theorem (3 Punkte)

$$\begin{aligned}
\frac{d}{dt}\langle \hat{p} \rangle &= \frac{d}{dt}\langle \psi | \hat{p} | \psi \rangle = \langle \dot{\psi} | \hat{p} | \psi \rangle + \underbrace{\langle \psi | \dot{\hat{p}} | \psi \rangle}_{=0} + \langle \psi | \hat{p} | \dot{\psi} \rangle \\
&\overset{SG}{=} \frac{i}{\hbar}\langle \psi | \hat{H}\hat{p} | \psi \rangle - \frac{i}{\hbar}\langle \psi | \hat{p}\hat{H} | \psi \rangle \\
&= \frac{i}{\hbar}\langle \psi | [\hat{H}, \hat{p}] | \psi \rangle
\end{aligned}$$

Mit $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ ergibt sich für den Kommutator:

$$\begin{aligned}
\left[\hat{H}, \hat{p}\right] &= \underbrace{\left[\frac{\hat{p}^2}{2m}, \hat{p}\right]}_{=0} + [V(\hat{x}), \hat{p}] \\
&= i\hbar \frac{\partial V(x)}{\partial x}
\end{aligned}$$

Und somit:

$$\begin{aligned} \frac{d}{dt}\langle\hat{p}\rangle &= \frac{i}{\hbar}\left\langle\psi\left|i\hbar\frac{\partial V(x)}{\partial x}\right|\psi\right\rangle \\ &= -\left\langle\frac{\partial V(x)}{x}\right\rangle \end{aligned}$$

□