



**Übungen zur Theoretischen Physik III für Lehramtskandidaten
 (Quantenmechanik+Statistische Physik)
 SS 2010, Blatt 1**

1. Aufgabe: Wellenfunktion (3+3 Punkte)

Einige Regeln für komplexe Konjugation:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$\overline{\varphi(z)} = \varphi(\overline{z})$ für jede holomorphe Funktion φ mit $\varphi(x) \in \mathbb{R}$ für $x \in \mathbb{R}$

a)

$$\begin{aligned} \psi(x, t) &= \int_{-\infty}^{\infty} dk g(k) e^{i(kx - \hbar \frac{k^2}{2m} t)} \\ &= \int_{-\infty}^{\infty} dk A e^{-\frac{a^2}{2} k^2} e^{i(kx - \hbar \frac{k^2}{2m} t)} \\ &= A \int_{-\infty}^{\infty} dk e^{ikx} e^{-\left(\frac{a^2}{2} + i \frac{\hbar t}{2m}\right) k^2} \\ &= A \frac{\sqrt{2\pi}}{a} \left(1 + i \frac{\hbar t}{ma^2}\right)^{-1/2} \exp\left[-\frac{x^2}{2a^2 \left(1 + i \frac{\hbar t}{ma^2}\right)}\right] \end{aligned}$$

Komplexe Konjugation der Wellenfunktion ergibt:

$$\begin{aligned} \psi^*(x, t) &= A^* \frac{\sqrt{2\pi}}{a} \left(\left(1 + i \frac{\hbar t}{ma^2}\right)^{-1/2}\right)^* \left(\exp\left[-\frac{x^2}{2a^2 \left(1 + i \frac{\hbar t}{ma^2}\right)}\right]\right)^* \\ &= A^* \frac{\sqrt{2\pi}}{a} \left(1 - i \frac{\hbar t}{ma^2}\right)^{-1/2} \exp\left[-\frac{x^2}{2a^2 \left(1 - i \frac{\hbar t}{ma^2}\right)}\right] \end{aligned}$$

$$\begin{aligned} |\psi(x, t)|^2 &= \psi(x, t) \psi^*(x, t) \\ &= |A|^2 \frac{2\pi}{a^2} \left[\left(1 + i \frac{\hbar t}{ma^2}\right) \left(1 - i \frac{\hbar t}{ma^2}\right)\right]^{-1/2} \exp\left[-\frac{x^2}{2a^2} \left(\frac{1}{1 + i \frac{\hbar t}{ma^2}} + \frac{1}{1 - i \frac{\hbar t}{ma^2}}\right)\right] \\ &= |A|^2 \frac{2\pi}{a^2} \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]^{-1/2} \exp\left[-\frac{x^2}{a^2 \left(1 + \left(\frac{\hbar t}{ma^2}\right)^2\right)}\right] \end{aligned}$$

Normierung:

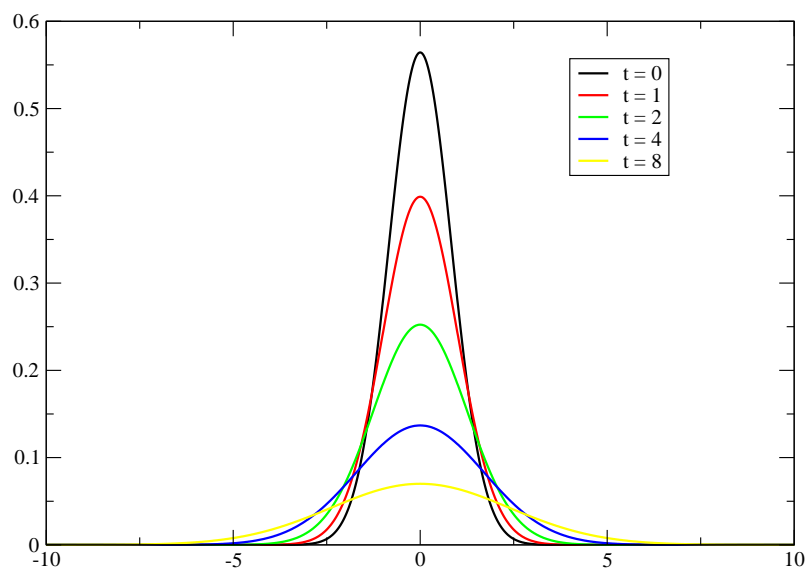
$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} dx |\psi(x, t)|^2 = |A|^2 \frac{2\pi}{a^2} \left[1 + \left(\frac{\hbar t}{ma^2} \right)^2 \right]^{-1/2} \left(2\pi \frac{a^2}{2} \left(1 + \left(\frac{\hbar t}{ma^2} \right)^2 \right) \right)^{1/2} \\ &= |A|^2 \frac{2\pi^{3/2}}{a} \\ \Rightarrow A &= \left(\frac{a}{2\pi^{3/2}} \right)^{1/2} \end{aligned}$$

Damit lautet die Wellenfunktion:

$$\psi(x, t) = a^{-1/2} \pi^{-1/4} \left(1 + i \frac{\hbar t}{ma^2} \right)^{-1/2} \exp \left[-\frac{x^2}{2a^2 \left(1 + i \frac{\hbar t}{ma^2} \right)} \right]$$

b)

$$|\psi(x, t)|^2 = \frac{1}{a\sqrt{\pi}} \left[1 + \left(\frac{\hbar t}{ma^2} \right)^2 \right]^{-1/2} \exp \left[-\frac{x^2}{a^2 \left(1 + \left(\frac{\hbar t}{ma^2} \right)^2 \right)} \right]$$



Setze

$$\sigma^2 := \frac{a^2}{2} \left(1 + \left(\frac{\hbar t}{ma^2} \right)^2 \right).$$

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} dx \psi^*(x, t) x \psi(x, t) \\
&= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} dx x \exp\left[-\frac{x^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} dx \frac{d}{dx} \left[-\sigma^2 \exp\left[-\frac{x^2}{2\sigma^2}\right] \right] \\
&= -\frac{\sigma}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right] \Big|_{-\infty}^{+\infty} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} dx \psi^*(x, t) x^2 \psi(x, t) \\
&= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} dx x \cdot x \exp\left[-\frac{x^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi\sigma}} \left(x (-\sigma^2) \exp\left[-\frac{x^2}{2\sigma^2}\right] \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} dx (-\sigma^2) \exp\left[-\frac{x^2}{2\sigma^2}\right] \right) \\
&= \sigma^2
\end{aligned}$$

$$\begin{aligned}
\Delta x &= \sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sqrt{\langle x^2 \rangle} \\
&= \sigma \\
&= \frac{a^2}{2} \left(1 + \left(\frac{\hbar t}{ma^2} \right)^2 \right)
\end{aligned}$$

2. Aufgabe: Ehrenfest'sche Theorem (3 Punkte)

$$\begin{aligned}
\frac{d}{dt} \langle \hat{p} \rangle &= \frac{d}{dt} \langle \psi | \hat{p} | \psi \rangle = \langle \dot{\psi} | \hat{p} | \psi \rangle + \underbrace{\langle \psi | \dot{\hat{p}} | \psi \rangle}_{=0} + \langle \psi | \hat{p} | \dot{\psi} \rangle \\
&\stackrel{SG}{=} \frac{i}{\hbar} \langle \psi | \hat{H} \hat{p} | \psi \rangle - \frac{i}{\hbar} \langle \psi | \hat{p} \hat{H} | \psi \rangle \\
&= \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{p}] | \psi \rangle
\end{aligned}$$

Mit $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ ergibt sich für den Kommutator:

$$\begin{aligned}
[\hat{H}, \hat{p}] &= \underbrace{\left[\frac{\hat{p}^2}{2m}, \hat{p} \right]}_{=0} + [V(\hat{x}), \hat{p}] \\
&= i\hbar \frac{\partial V(x)}{\partial x}
\end{aligned}$$

Und somit:

$$\begin{aligned}\frac{d}{dt}\langle\hat{p}\rangle &= \frac{i}{\hbar}\left\langle\psi\left|i\hbar\frac{\partial V(x)}{\partial x}\right|\psi\right\rangle \\ &= -\left\langle\frac{\partial V(x)}{\partial x}\right\rangle\end{aligned}$$

□